

System size stochastic resonance in driven finite arrays of coupled bistable elements

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Abstract

The global response to weak time periodic forces of an array of noisy, coupled nonlinear systems might show a nonmonotonic dependence on the number of units in the array. This effect has been termed system size stochastic resonance. In this paper, we analyze the nonmonotonic dependence on the system size of the signal-to-noise ratio of a collective variable characterizing a finite array of one-dimensional globally coupled bistable elements. By contrast with the conventional non-monotonic dependence with the strength of the noise (stochastic resonance), system size stochastic resonance is found to be restricted to some regions in parameter space.

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I. INTRODUCTION

In some nonlinear systems, noise can be used to improve the response of the system to a weak external signal. In the last two decades, this rather counterintuitive effect known as Stochastic Resonance (SR), has been massively studied, both experimentally and theoretically, due to its potential interest to a variety of areas of science [1] and technology [2]. A large majority of the experimental as well as theoretical studies have dealt mainly with the response of single systems to weak external forcings. Nonetheless, the response of complex systems like arrays composed of many coupled units has also been considered [3, 4, 5, 6, 7, 8, 9, 10, 11, 12].

The study of complex systems opens new possibilities with respect to those observed in single unit systems. Besides the usual noise strength, the variation of other system parameters can give rise to enhanced, nonmonotonic behaviors of some magnitudes characterizing the system response to a weak externally applied time periodic force. Indeed, nonmonotonic behaviors of pertinent quantifiers with the coupling strength parameter of the array have been reported in the literature [3, 4, 5, 6, 7, 8, 9, 10, 11].

The noise enhanced nonmonotonic response of driven systems with the number of units has been termed system size stochastic resonance (SSSR). It has been studied in several contexts. Several authors have considered the dependence of system properties on the number of elementary units N in complex arrays of bistable elements [12, 13, 14]. The dependence on the number of ion channels of the ion concentrations along cell membranes in biological Hodgkin-Huxley type models has been studied in [15, 16]. In [17], the authors analyze the optimization of neuron firing events in coupled excitable neuron units as the number of coupled elements is varied. The phenomenon of SSSR has also been discussed within the context of modelling opinion formation in social collectivities [18].

In this paper, we address the phenomenon of SSSR for arrays of globally coupled noisy bistable units. We consider that the collective response of the whole complex system is well represented by a single random variable. Its dynamics is clearly constructed from the dynamics of the constituent elements of the array but it shows collective emergent properties. In Section II we introduce a model in terms of the individual units and the interactions among them, and we define the collective variable and the quantifier that we will use to characterize the phenomenon. Next, in Section III, we discuss the possibility of a reduced description of

the collective variable dynamics in terms of an effective Langevin equation, as put forward by Pikovsky et al. in [12]. In Section IV, we address the problem with numerical simulation tools and discuss the results obtained. The paper ends with some conclusions.

II. THE MODEL

The model considered in this work is the same as the one introduced by Desai and Zwanzig in their influential paper [19], except that we add a time periodic force to the Langevin dynamics of each degree of freedom. This model is also the one considered in [12]. The model consists of a set of N identical bistable units, each of them characterized by a variable $x_i(t)$ ($i = 1, \dots, N$) satisfying stochastic evolution equations (in dimensionless form) of the type

$$\dot{x}_i = x_i - x_i^3 + \frac{\theta}{N} \sum_{j=1}^N (x_j - x_i) + \sqrt{2D} \xi_i(t) + F(t), \quad (1)$$

where θ is a coupling parameter and the term $\xi_i(t)$ represents a white noise with zero average and $\langle \xi_i(t) \xi_j(s) \rangle = \delta_{ij} \delta(t - s)$. The external driving force is periodic in t , $F(t) = F(t + T)$. An alternative formulation of the dynamics is in terms of the Fokker-Planck equation for the joint probability density $f_N(x_1, \dots, x_N, t)$,

$$\frac{\partial f_N}{\partial t} = \sum_{i=1}^N \frac{\partial}{\partial x_i} \left(\frac{\partial U}{\partial x_i} f_N \right) + D \sum_{i=1}^N \frac{\partial^2 f_N}{\partial x_i^2}, \quad (2)$$

where U is a time dependent potential energy relief,

$$\begin{aligned} U(x_1, \dots, x_N, t) = & \sum_{i=1}^N V(x_i) + \frac{\theta}{4N} \sum_{i=1}^N \sum_{j=1}^N (x_i - x_j)^2 \\ & + F(t) \sum_{i=1}^N x_i \end{aligned} \quad (3)$$

with

$$V(x) = \frac{x^4}{4} - \frac{x^2}{2}. \quad (4)$$

We are interested in the properties of a collective variable, $S(t)$, defined as

$$S(t) = \frac{1}{N} \sum_{j=1}^N x_j(t), \quad (5)$$

characterizing the chain as a whole. Even though the set $x_i(t)$ is a non-stationary N -dimensional Markovian process, $S(t)$ is not, in general, a 1-dimensional Markovian process.

Several magnitudes showing enhanced nonmonotonic behaviors as a system parameter is varied have been used as SR quantifiers [1]. In this work, we will use the signal-to-noise ratio of the collective variable R_{out} as the relevant quantifier. In order to evaluate it, we need the knowledge of the one-time correlation function of the collective variable defined as

$$C(\tau) = \frac{1}{T} \int_0^T dt \langle S(t)S(t+\tau) \rangle_\infty, \quad (6)$$

The notation $\langle \dots \rangle$ indicates an average over the noise realizations and the subindex ∞ indicates the long time limit of the noise average, i. e., its value after waiting for t long enough that transients have died out. As shown in our previous work [10], it is convenient to split $C(\tau)$ into two parts,

$$C(\tau) = C_{\text{coh}}(\tau) + C_{\text{incoh}}(\tau). \quad (7)$$

The coherent part, $C_{\text{coh}}(\tau)$, given by

$$C_{\text{coh}}(\tau) = \frac{1}{T} \int_0^T dt \langle S(t) \rangle_\infty \langle S(t+\tau) \rangle_\infty, \quad (8)$$

is periodic in τ with the period of the driving force. The incoherent part, $C_{\text{incoh}}(\tau)$ arising from the fluctuations of the output $S(t)$ around its average value, decays to zero as τ increases.

The quantifier R_{out} is defined as

$$R_{\text{out}} = \lim_{\epsilon \rightarrow 0^+} \frac{\int_{\Omega-\epsilon}^{\Omega+\epsilon} d\omega \tilde{C}(\omega)}{\tilde{C}_{\text{incoh}}(\Omega)} = \frac{\tilde{C}_{\text{coh}}(\Omega)}{\tilde{C}_{\text{incoh}}(\Omega)}, \quad (9)$$

where Ω is the fundamental frequency of the driving force $F(t)$, $\tilde{C}_{\text{coh}}(\Omega)$ is the corresponding Fourier coefficient in the Fourier series expansion of $C_{\text{coh}}(\tau)$, and $\tilde{C}_{\text{incoh}}(\Omega)$ is the Fourier transform at frequency Ω of $C_{\text{incoh}}(\tau)$.

III. APPROXIMATE DYNAMICS FOR $S(t)$

The dynamics of the collective variable follows from its definition and the dynamics of the individual degrees of freedom. But, as $S(t)$ is not Markovian in general, there is no reason why it should satisfy a 1-dimensional closed Langevin equation with effective drift and diffusion coefficients. For the same reason, there is no guarantee that its associated probability density $P(s, t)$ would satisfy a Fokker-Planck equation. On the other hand,

having an approximate description in terms of a 1-dimensional stochastic process, would certainly be a helpful tool for the understanding of the dynamical behaviors. It is then not surprising that several approximate descriptions in a reduced 1-dimensional space for the collective variable have been proposed in the literature [12, 20]. In [12], Pikovsky et al. used the Gaussian truncation of an infinite hierarchy of equations for the cumulants and a slaving principle to construct an effective 1-dimensional Langevin equation. In the absence of external driving, its explicit form is

$$\dot{S} = aS - bS^3 + \sqrt{\frac{2D}{N}}\chi(t) \quad (10)$$

with $\langle \chi(t)\chi(s) \rangle = \delta(t-s)$ and the coefficients a and b given by

$$\begin{aligned} a &= 1 + 0.5(\theta - 1) - 0.5\sqrt{(\theta - 1)^2 + 12D}; \\ b &= \frac{4a}{2 - \theta + \sqrt{(2 + \theta)^2 - 24D}}. \end{aligned} \quad (11)$$

Unfortunately, the coefficient b does not exist for all ranges of the θ and D values. In Fig. (1) the continuous line represents $(2 + \theta)^2 - 24D = 0$ containing the function appearing in the definition of b . This line separates regions where the effective Langevin equation in Eq. (10) either exists or not. Below that line, the b coefficient is real and the Langevin equation, Eq. (10), exists, while for points above the line b is complex, and therefore, the effective Langevin equation is not valid.

In general, R_{out} can not be evaluated analytically. But if we accept that $S(t)$ satisfies a Langevin equation like Eq. (10) with an extra term corresponding to the driving force added to its right hand side and this extra term is very weak, then a linear response theory might provide a valid approximation to analyze the system behavior. The problem is then similar to that of SR in a single bistable potential and we can apply well known arguments to obtain analytical approximations [12, 21] for R_{out} within the limits of linear response theory. This is the strategy followed by Pikovsky and coworkers in [12]. They find SSSR for weak amplitude driving forces and some ranges of noise strength and coupling parameter values. Due to the difficulties with Eq. (10) discussed above, it is not clear whether the phenomenon exists for all parameter values or it is restricted to some particular regions of the parameter space. It is also interesting to relate the possible SSSR to the stochastic resonant effects in coupled arrays reported in other works [9, 10, 11] when the noise or the coupling constant are varied.

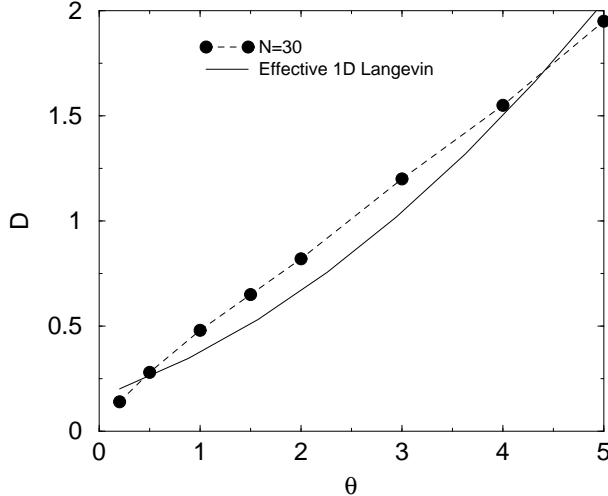


FIG. 1: The solid line represents $(2 + \theta)^2 - 24D = 0$. For points above it, the coefficient b is a complex number and the effective Langevin equation (10) is not valid. The dotted line joining black dots marks a numerically obtained transition line such that, above this line, the equilibrium distribution of the global variable is always monomodal and below it $P^{\text{eq}}(s)$ is multimodal (see the final paragraph of Section IV). The data depicted were obtained for a system with $N = 30$ units, but those values are very independent of the system size as long as it remains finite.

IV. NUMERICAL RESULTS

The lack of a reliable effective Langevin approximation valid for all regions of parameter space forces us to consider numerical simulations. Following the procedure indicated in [22], we have numerically solved the Langevin equations for $x_i(t)$ in Eq. (1) for very many noise realizations. The numerical solution of the N -dimensional process is used to collect information about $S(t)$ and construct the magnitudes that we need. In particular we will estimate numerically the long-time limit of the first two cumulant moments, the correlation function with its coherent and incoherent parts and histograms to estimate the probability density. The estimation of R_{out} is then a matter of numerically performing the quadratures indicated in its definition (see Eq. (9)).

Let us consider a rectangular driving force: $F(t) = A$ ($F(t) = -A$) if $t \in [nT/2, (n + 1)T/2)$ with n even (odd). In what follows, we will always take the amplitude $A = 0.05$ and the fundamental frequency $\Omega = 0.05$. When such a force is applied to a single isolated noisy bistable, the response of the system is very well described by a linear response function, as

the dynamical effects arising from the distortion of the bistable potential induced by the driving are within the limits of a small perturbation theory.

A. Strong coupling

First we study the case of parameter values within the range considered in [12]. In Fig. (2), the black circles correspond to the numerically obtained values of R_{out} as the number of particles N of the array is varied and $\theta = 5.5$, $D = 1$. The existence of SSSR is evident, in consonance with the results reported in [12]. It should also be pointed out that the values of R_{out} are rather small, even for arrays with a size where R_{out} reaches its peak value. To understand why, we have analyzed the time behavior of the first two cumulants of the collective variable as well as the incoherent part of the fluctuations about its average. In Fig. (3b) we depict the time evolution of the first two cumulant moments for the parameter values $\theta = 5.5$, $D = 1$. Even though the amplitude of the average output is about five times bigger than that of the driving force, the output signal is very noisy with a second cumulant much larger than the noise strength $(2D)/30 \sim 0.066$ associated to the effective Langevin equation, Eq. (10). The incoherent fluctuations of the collective variable are depicted in Fig. (4b). They are large and long-lasting. Consequently, the numerator in the fraction defining R_{out} (see Eq. (9)) should not be very large, while its denominator has a substantial value. The smallness of the R_{out} values is not surprising.

The results of Pikovsky et al. [12] rely on the validity of the linear response approximation. Within this limit, one assumes that the incoherent part of the output correlation function in a driven system can be safely approximated by the equilibrium correlation function of the same system in the absence of driving force. It is clear from Fig. (4b) that such an approximation is not too bad. Then, for the parameter values $\theta = 5.5$, $D = 1$, the linear response function can be used and as this function has a nonmonotonic behavior with N , one finds SSSR.

Let us now analyze what happens if we increase the noise strength to $D = 2.5$, while keeping all the other parameter values fixed. The black squares in Fig. (2) indicate that R_{out} increases monotonically with the size of the system. Thus, there is no SSSR for this noise strength even though the coupling parameter is still strong. Note that for these noise strength and coupling parameter values, the effective Langevin equation in Eq. (10) does not

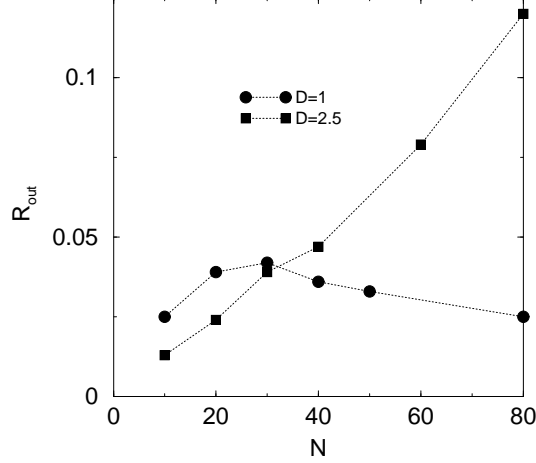


FIG. 2: The signal-to-noise ratio of the collective output R_{out} as a function of N for two different values of the noise intensity D . In both cases $\theta = 5.5$. The driving amplitude is $A = 0.05$ and the fundamental frequency $\Omega = 0.05$.

exist. On the other hand, the numerical results depicted in Figs. (3a) and (4a) show that the second cumulant is reduced with respect to its value for $D = 1$, and that the incoherent part of the fluctuations in the driven system matches well the equilibrium fluctuations. Taking also into account the smallness of the driving amplitude, a linear response approximation should still provide a reliable description of the response of the system for large noises and strong couplings, even though SSSR does not exist in that region.

It is worth noting that a nonmonotonic behavior of R_{out} with the noise strength D , typical of stochastic resonance, does exist for strong coupling. An example is depicted in Fig. (5) for a system of $N = 30$ elements with coupling strength $\theta = 5.5$, driven by a weak rectangular force with $A = 0.05$ and $\Omega = 0.05$. As expected, the values of R_{out} are quite small.

B. Weak coupling

Let us now consider cases with a much weaker coupling parameter. We will take $\theta = 0.5$. We will consider two values of the noise strength: $D = 1$ and $D = 0.2$. In Fig. (6) we depict the behavior of R_{out} with the system size. Clearly, there is no SSSR as the value of R_{out} increases with N for both noise values.

The behavior of the first two cumulant moments of the collective variable for a driven system with $N = 30$ units and coupling parameter value $\theta = 0.5$ is depicted in Fig. (7)

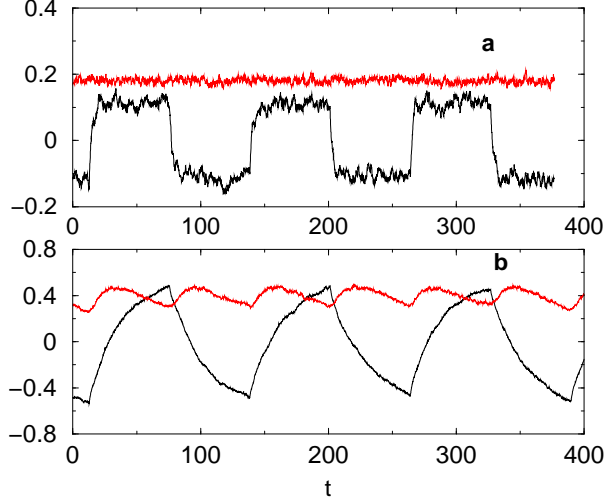


FIG. 3: (Color online) Time evolution of the first two cumulant moments of $S(t)$ in a system with $N = 30$ strongly coupled ($\theta = 5.5$) units driven by a rectangular force with $A = 0.05$ and $\Omega = 0.05$, for $D = 2.5$ (a) and $D = 1$ (b).

for two values of the noise strength: $D = 0.2$ in the lower panel, and $D = 1$ in the upper one. It is to be noted the large increase of the output amplitude relative to the weak driving amplitude for the $D = 0.2$ case, while in the higher noise situation, the amplitude amplification is small. The second cumulant plots indicate that the output noise level is also very much increased in the low D case, while it is kept very small in for the higher noise.

The time dependence of the incoherent part of the correlation function of the collective variable for the driven system and the equilibrium correlation function of the same variable are shown in Fig. (8), for $N = 30$, $\theta = 0.5$, $A = 0.05$, $\Omega = 0.05$ and two noise values. For the higher noise value, $D = 1$, the incoherent fluctuations in the driven system are practically identical to the equilibrium correlation function. This feature indicates that a linear response theory should then be an excellent approximation to analyze the response of the system to weak driving forces, when the noise strength is large even in a weak coupling situation. On the other hand, for the low noise strength case $D = 0.2$, the situation is different. The time behavior of $C_{\text{incoh}}(t)$ is very different from the time behavior of $C_{\text{eq}}(t)$. The correlations decay much faster in the driven system than in the absence of driving and from a smaller initial value. This indicates that the driving field, albeit weak, introduces a very strong distortion of the fluctuation dynamics of the collective variable during each cycle of the external driving, thus rendering inadequate the ideas behind a linear response

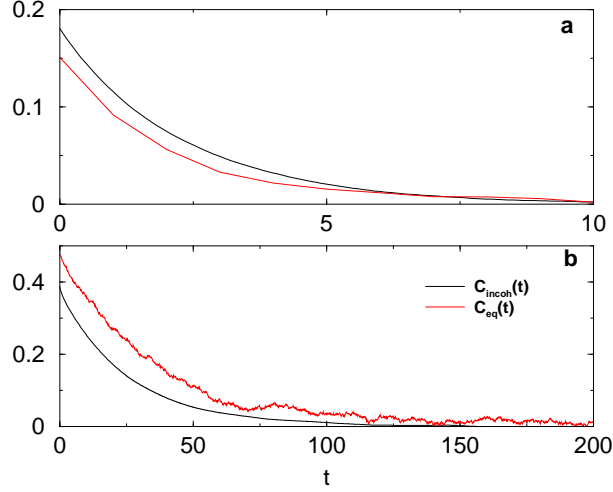


FIG. 4: (Color online) Time evolution of the incoherent part of the correlation function (black line) and the equilibrium time correlation function (red line) of the collective variable in a system with $N = 30$ strongly coupled ($\theta = 5.5$) units for $D = 2.5$ (a), and $D = 1$ (b). The driving force is rectangular with $A = 0.05$ and $\Omega = 0.05$.

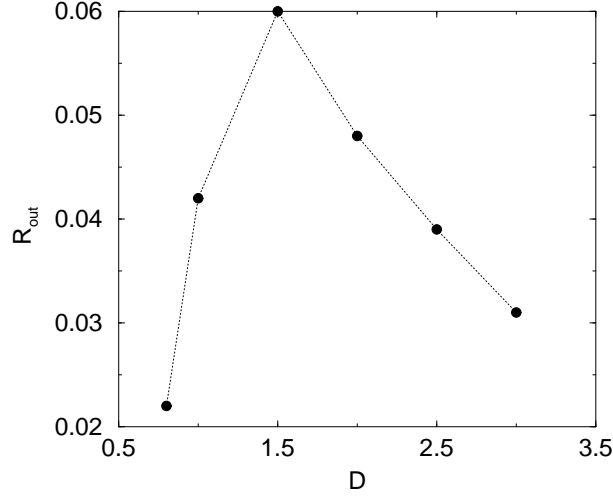


FIG. 5: Non-monotonic behavior of R_{out} with the noise strength D for a system with $N = 30$ units with global coupling strength $\theta = 5.5$. The driving force is rectangular with $A = 0.05$ and $\Omega = 0.05$.

approximation. This fact has been indeed analyzed previously by us in connection with SR in arrays of interacting particles [23, 24]. We have referred to this regime as a non-linear SR regime, characterized by the control of the fluctuations by the external driving.

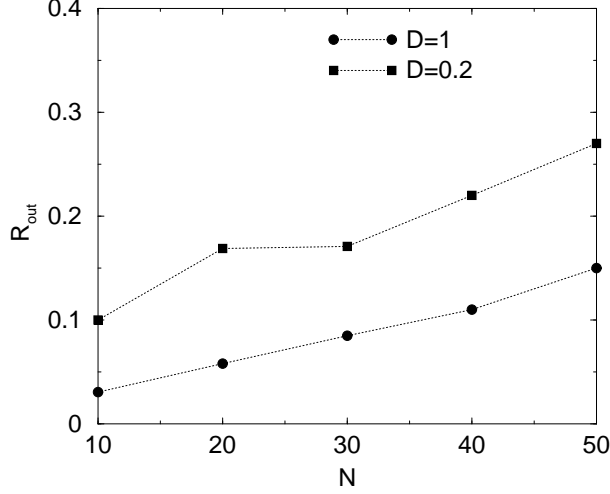


FIG. 6: Monotonic behavior of R_{out} with the number of units N . The data corresponds to $\theta = 0.5$ and two different values of D . The driving force is rectangular with $A = 0.05$ and $\Omega = 0.05$.

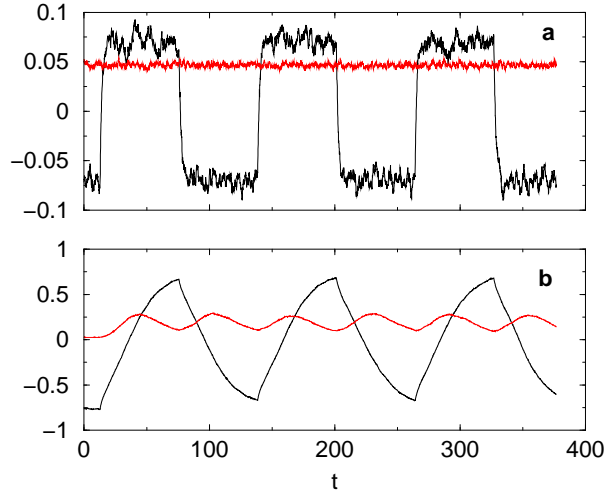


FIG. 7: (Color online) Time evolution of the first two cumulant moments of $S(t)$ in a system with $N = 30$ weakly coupled ($\theta = 0.5$) units driven by a rectangular force with $A = 0.05$ and $\Omega = 0.05$, for $D = 1$ (a) and $D = 0.2$ (b)

In Fig. (9) we depict the nonmonotonic behavior of R_{out} with D for an array of weakly coupled ($\theta = 0.5$) $N = 30$ units, driven by a rectangular force with $A = 0.05$ and $\Omega = 0.05$. The values of R_{out} are substantially larger than the ones obtained for the stronger coupling constant (compare with Fig. 5).

To help rationalize the results above, it is useful to consider the equilibrium distribution

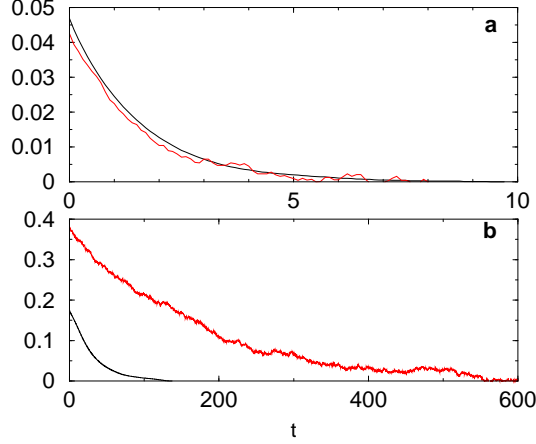


FIG. 8: (Color online) Time evolution of the incoherent part of the correlation function (black line) and the equilibrium time correlation function (red line) of the collective variable in a system with $N = 30$ weakly coupled ($\theta = 0.5$) units for $D = 1$ (a), and $D = 0.2$ (b). The driving force is rectangular with $A = 0.05$ and $\Omega = 0.05$.

function of the collective variable $P^{\text{eq}}(s) = \langle \delta(s - S(t)) \rangle$. This function can be estimated by constructing histograms from the long time results of the numerical simulations of Eq. (1) with $F(t) = 0$. The data reveals the existence of a transition line separating different regions in the D - θ plane. The line joining the black circles in Fig. (1) corresponds to the transition line for a system with $N = 30$ particles. The location of the line for systems with other number of particles do not differ substantially from the one plotted here. For parameter values corresponding to points above the line, the equilibrium distribution is always monomodal and centered around $s = 0$, while for points below the line, it is bimodal with one minimum at $s = 0$ and two maxima at values $\pm s_0$. The location of the minima depends on the values of the system parameters. Thus for a fixed N and θ , as the noise value is increased, the equilibrium distribution of the system would change shape from bimodal to monomodal. The nonmonotonic behavior of R_{out} with D (SR) observed in both Figs. (5) (for strong coupling) and (9) (for weak coupling) is associated the change in the shape of the equilibrium distribution function. By contrast, SSSR is restricted to a reduced region of parameter space involving rather high values of the coupling strength and relatively low values of the noise strength without involving a change in shape of $P^{\text{eq}}(s)$. SSSR is then connected to conditions such that a bimodal equilibrium distribution is only slightly distorted by the driving field. Note that even though a linear response approach might yield a reliable

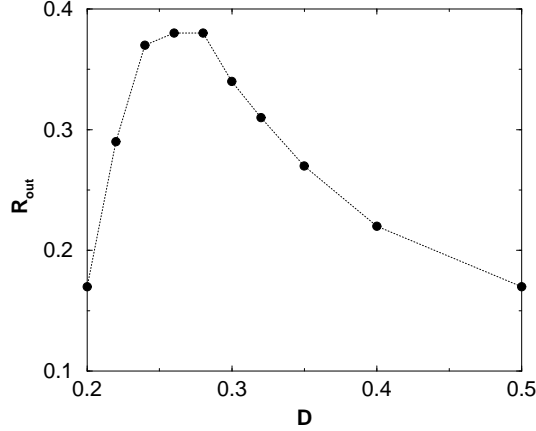


FIG. 9: Non-monotonic behavior of R_{out} with D for $N = 30$ weakly coupled ($\theta = 0.5$) units. The driving force is rectangular with $A = 0.05$ and $\Omega = 0.05$.

description of the dynamics, as it happens for points above the transition line, the existence of SSSR is not guaranteed. In this sense, SSSR does not seem to be such a general effect as the usual SR nonmonotonic behavior with the noise strength.

V. CONCLUSIONS

In this work, we have analyzed the response of globally coupled arrays of noisy bistable systems driven by weak time periodic forces. We focus on the dynamics of a 1-dimensional collective variable characterizing the array as a whole. The lack of an adequate Langevin equation describing the reduced dynamics of the collective variable, prompts us to use numerical simulations involving all the individual dynamics.

Our results indicate that SSSR exists only in limited regions of parameter space: for relatively low noise values and strongly coupled arrays. By looking at the behavior of the fluctuations in those regions, we find that the system response is adequately well described by a linear response approximation. Furthermore, the reduced dynamics can be well described by an effective Langevin equation.

Even for strongly coupled arrays, SSSR does not exist when the noise values are large. The effective Langevin equation proposed in [12] ceases to be valid. On the other hand, the equilibrium probability of the collective variable is, in this region of parameter values, monomodal. Thus, the effect of a weak applied external force on the collective dynamics

should still well described by a linear response function which has a monotonic behavior with N .

For weak coupling, the SSSR disappears. Regardless of the noise strength, R_{out} monotonically increases with the number of particles in the array. For large noise values, $P^{\text{eq}}(s)$ is monomodal. Thus, a linear response approximation should be a reliable description of the system response. On the other hand, for low noise values, the influence of the applied force on the fluctuations dynamics is so intense that the system operates in a non-linear regime.

Our results demonstrate that in the strong coupling regime, there is a typical SR behavior with the noise strength D , when the system size N and the coupling constant θ are kept fixed. The R_{out} values are, nonetheless, very small. This is in sharp contrast with the large enhancement of the SR effects observed in the weak coupling regime.

The construction of effective reduced 1-dimensional dynamics for the relevant variable is very useful. The approaches followed in [12, 20] are unfortunately limited to restricted regions of parameter space. We are currently working on obtaining approximate Langevin equations for the collective variable in driven systems valid for all regions in parameter space, along the lines initiated in [25].

Acknowledgments

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